Unit 06

ALGEBRAIC MANIPULATION

Highest Common Factor (H.C.F.)

If two or more algebraic expressions are given then their common factor of highest power is called the H.C.F of the expressions.

Least Common Multiple (L.C.M)

If an algebraic expression $p(x)$ is exactly divisible by two or more expressions, then p(x) is called the Common Multiple of the given expressions. The Least Common Multiple (L.C.M) is the product of common factors together with non-common factors of the given expressions.

Finding H.C.F

We can find H.C.F of given expressions by the following two methods.

(i) By Factorization

(ii) By division

H.C.F. by Factorization

Example

Find the H.C.F of the following polynomials.

$$x^2-4, x^2+4x+4, 2x^2+x-6$$

Solution

By factorization,

$$x^2-4=(x+2)(x-2)$$

$$x^2+4x+4=(x+2)^2=(x+2)(x+2)$$

$$2x^2+x-6=2x^2+4x-3x-6=2x(x+2)-3(x+2)$$
$$=(x+2)(2x-3)$$

Common factors $= x+2$

H.C.F $= x+2$

H.C.F. by Division

Example

Use division method to find the H.C.F. of the polynomials

$$p(x)=x^3-7x^2+14x-8 \quad and$$
$$q(x)=x^3-7x+6$$

Solution

$$\begin{array}{r|l} & \quad\quad 1 \\ \hline x^3-7x+6 & x^3-7x^2+14x-8 \\ & \underset{-}{+x^3} \quad\quad \underset{+}{-7x} \ \underset{-}{+6} \\ \hline & -7x^2+21x-14 \end{array}$$

Here the remainder can be factorized as

$$-7x^2+21x-14=-7(x^2-3x+2)$$

We ignore -7 because it is not common to both the given polynomials and consider x^2-3x+2.

$$\begin{array}{r|l} & \quad x+3 \\ \hline x^2-3x+2 & x^3+0x^2-7x+6 \\ & \underset{-}{+x^3}\underset{+}{-3x^2}\ \underset{-}{+2x} \\ \hline & 3x^2-9x+6 \\ & \underset{-}{3x^2}\underset{+}{-9x}\ \underset{-}{+6} \\ \hline & \quad 0 \end{array}$$

Hence H.C. F of $p(x)\, and\, q(x)$ is x^2-3x+2

Example

Find the L.C.M of $p(x)=12(x^3-y^3)\, and\, q(x)=8(x^3-xy^2)$

Solution

By prime factorization of the given expressions, we have

$p(x)=12(x^3-y^3)=2^2\times3\times(x-y)(x^2+xy+y^2)$ and

$q(x)=8(x^3-xy^2)=8x(x^2-y^2)=2^3x(x+y)(x-y)$ Hence L.C.M. of $p(x)$ and $q(x)$,

$2^3\times3\times x(x+y)(x-y)(x^2+xy+y^2)=24x(x+y)(x^3-y^3)$

Relation between H.C.F and L.C.M

Example

By factorization, find (i) H.C.F (ii) L.C.M of $p(x)=12(x^5-x^4)$ and $q(x)=8(x^4-3x^3+3x^2)$. Establish a relation between $p(x)$, $q(x)$ and H.C.F and L.C.M of the expressions $p(x)$ and $q(x)$.

Solution

Firstly, let us factorize completely the given expressions $p(x)$ and $q(x)$ into irreducible factors. We have

$p(x)=12(x^5-x^4)=12x^4(x-1)=2^2\times3\times x^4(x-1)$ and

$q(x) \quad = \quad 8(x^4-3x^3+2x^2)=8x^2(x^2-3x+2)=2^3x^2(x-1)(x-2)$

H.C.F. of $p(x)$ and $q(x) \quad = \quad 2^2x^2(x-1)=4x^2(x-1)$

L.C.M of $p(x)$ and $q(x) \quad = \quad 2^3\times3\times x^4(x-1)(x-2)$

Now $p(x)\, q(x) = 12\, x^4(x-1)\times 8\, x^2(x-1)(x-2)$

$= 96\, x^6(x-1)^2(x-2)$ ....(i)

and (L.C.M) (H.C.F)

$= [2^3\times3\times x^4(x-1)(x-2)]\,[4\,x^2(x-1)]$

$= [24\,x^4(x-1)(x-2)]\,[4\,x^2(x-1)]$

$= 96\,x^4(x-1)^2(x-2)$ .... (ii)

From (i) and (ii)

L.C.M x H.C.F = P(x) x q(x)

Note

(1) $\text{L.C.M}=\dfrac{p(x)\times q(x)}{\text{H.C.F}}$ or

$\text{H.C.F}=\dfrac{p(x)\times q(x)}{\text{L.C.M}}$

(2) If L.C.M, H.C.F and one of $p(x)$ or $q(x)$ are known, then

$$p(x) = \frac{\text{L.C.M}\times\text{H.C.F}}{q(x)}$$

$$q(x) = \frac{\text{L.C.M}\times\text{H.C.F}}{p(x)}$$

Example

Find H.C.F of the polynomials,

$$p(x) \quad = \quad 20(2x^3+3x^2-2x)$$

$$q(x) \quad = \quad 9(5x^4+40x)$$

Then using the above formula (I) find the L.C.M of $p(x)$ and $q(x)$.

Solution

We have

$p(x) = 20(2x^3+3x^2-2x) = 20x(2x^2+3x-2)$

$= 20x(2x^2+4x-x-2) = 20x[2x(x+2)-(x+2)] = 20x(x+2)(2x-1) = 2^2 \times 5 \times x(x+2)(2x-1)$

$q(x) \quad = \quad 9(5x^4+40x) = 45x(x^3+8) = 45x\left[\left(x^3\right)+\left(2\right)^3\right]$

$= \quad 45x(x+2)(x^2-2x+4) = 5 \times 3^2 \times x(x+2)(x^2-2x+4)$ Thus H.C.F of $p(x)$ and $q(x)$ is:

$= \quad 5x(x+2)$

Now, using the formula $\quad$ L.C.M. $= \dfrac{p(x) \times q(x)}{\text{H.C.F}}$

We obtain

$$\text{L.C.M.} = \frac{2^2 \times 5 \times x(x+2)(2x-1) \times 5 \times 3^2 \times x(x+2)(x^2-2x+4)}{5x(x+2)}$$

$$= 4 \times 5 \times 9 \times x(x+2)(2x-1)(x^2-2x+4)$$

$$= 180x(x+2)(2x-1)(x^2-2x+4)$$

Example

Find the L.C.M of

$p(x) = 6x^3-7x^2-27x+8$ and

$q(x) = 6x^3+17x^2+9x-4$

Solution

We have, by long division,

$$\begin{array}{r|l} & \quad 1 \\ \hline 6x^3-7x^2-27x+8 & 6x^3+17x^2+9x-4 \\ & 6x^3-7x^2\ \ -27x+8 \\ & -\quad +\qquad +\quad - \\ \hline & \qquad 24x^2+36x-12 \end{array}$$

But the remainder $24x^2+36x-12$

$= 12(2x^2+3x-1)$

Thus, ignoring 12, we have

$$\begin{array}{r|l} & 3x-8 \\ \hline 2x^2+3x-1 & 6x^3-7x^2-27x+8 \\ & 6x^3+9x^2\ \ -3x \\ & -\quad -\qquad + \\ \hline & \quad -16x^2-24x+8 \\ & \quad -16x^2-24x+8 \\ & \quad +\qquad +\qquad - \\ \hline & \qquad\qquad 0 \end{array}$$

Hence H.C.F of $p(x)$ and $q(x)$ is

$= 2x^2+3x-1$

$$x^2+6x-27=x^2-3x+9x-27$$

$$=x(x-3)+9(x-3)$$

$$=(x-3)(x+9) \quad ....(ii)$$

$$2x^2-18=2(x^2-9)$$

$$=2\left[(x)^2-(3)^2\right]$$

$$=2(x+3)(x-3) \quad ....(iii)$$

From (i), (ii) and (iii)

Common factors = $(x-3)$

$HCF = x-3$

iii) $x^3-2x^2+x,\ x^2+2x-3,\ x^2+3x-4$

Sol: By factorization

$$x^3-2x^2+x=x(x^2-2x+1)$$

$$=x(x^2-x-x+1)$$

$$=x[x(x-1)-1(x-1)]$$

$$=x(x-1)(x-1) \quad ....(i)$$

$$x^2+2x-3=x^2-x+3x-3$$

$$=x(x-1)+3(x-1)$$

$$=(x-1)(x+3) \quad ....(ii)$$

$$x^2+3x-4=x^2-x+4x-4$$

$$=x(x-1)+4(x-1)$$

$$=(x-1)(x+4) \quad ....(iii)$$

From (i), (ii) and (iii)

Common factors: $x-1$

HCF $=x-1$

iv) $18(x^3+9x^2+8x),\ 24(x^2-3x+2)$

Sol: By factorization

$$18(x^3-9x^2+8x)=18x(x^2-9x+8)$$

$$=18x(x^2-x-8x+8)$$

$$=18x[x(x-1)-8(x-1)]$$

$$=2\times3\times3\ x(x-1)(x-8) \quad ....(i)$$

$$24(x^2-3x+2)=$$

$$24(x^2-x-2x+2)$$

$$=2\times2\times2\times3[x(x-1)-2(x-1)]$$

$$=2\times2\times2\times3(x-1)(x-2) ....(ii)$$

From (i) and (ii)

HCF $=2\times3(x-1)$

$=6(x-1)$

v) $36(3x^4+5x^3-2x^2),\ 54(27x^4-x)$

Sol: By factorization

$$36(3x^4+5x^3-2x^2)=36x^2(3x^2+5x-2)$$

$$=36x^2(3x^2+6x-x-2)$$

$$=36x^2[3x(x+2)-1(x+2)]$$

$$=2\times2\times3\times3\,x.x(x+2)(3x-1) \quad ....(i)$$

$$54(27x^4-x)=54x(27x^3-1)$$

$$=54x\left[(3x)^3-(1)^3\right]$$

$$=54x(3x-1)\left[(3x)^2+(3x)(1)+(1)^2\right]$$

$$=2\times3\times3\times3x(3x-1)(9x^2+3x+1) \quad .....(ii)$$

From (i) and (ii)

Common factors = $2,3,3,x,(3x-1)$

HCF = $2\times3\times3x(3x-1)$

$=18x(3x-1)$

Q3. Find the H.C.F of the following by division methal.

i) $p(x)=x^3+3x^2-16x+12,\ q(x)=x^3+x^2-10x+8$

Sol:
$$\begin{array}{r|l} & \quad 1 \\ \hline x^3+x^2-10x+8 & x^3+3x^2-16x+12 \\ & \underline{-x^3 \pm x^2 \mp 10x \pm 8} \\ & \quad 2x^2-6x+4 \end{array}$$

Dividing remainder by 2

x^2-3x+2

$$
\begin{array}{r}
x+4 \\
x^2-3x+2\big)\overline{\cancel{x^3}+x^2-10x+8} \\
\underline{-\cancel{x^3}\mp 3x^2\pm 2x} \\
\cancel{4x^2}-\cancel{12x}+\cancel{8} \\
\underline{\cancel{-4x^2}-\cancel{12x}\pm\cancel{8}} \\
0
\end{array}
$$

Hence HCF $= x^2-3x+2$

ii) $P(x)=x^4+x^3-2x^2+x-3,$

$q(x)=5x^3+3x^2-17x+6$

$$
\begin{array}{rl}
x+2 & \\
5x^3+3x^2-17x+6\big)\overline{x^4+x^3-2x^2+x-3} & \\
\underline{\times 5\qquad\qquad\qquad\qquad} & \text{(Multiplying by 5)} \\
\cancel{5x^4}+5x^3-10x^2+5x-15 & \\
\underline{\cancel{-5x^4}\pm 3x^3\mp 17x^2\pm 6x} & \\
2x^3+7x^2-x-15 & \\
\underline{\times 5\qquad\qquad\qquad\qquad} & \text{(Multiplying by 5)} \\
\cancel{10x^3}+35x^2-5x-75 & \\
\underline{\cancel{-10x^3}\pm 6x^2\mp 34x\pm 12} & \\
29x^2+29x-87 &
\end{array}
$$

Divided by 29

x^2+x-3

$$
\begin{array}{r}
5x-2 \\
x^2+x-3\big)\overline{\cancel{5x^3}+3x^2-17x+6} \\
\underline{\cancel{-5x^3}\pm 5x^2\mp 15x} \\
\cancel{-2x^2}-\cancel{2x}+\cancel{6} \\
\underline{\cancel{\mp 2x^2}\mp\cancel{2x}\pm\cancel{6}} \\
0
\end{array}
$$

Hence H.C.F $= x^2+x-3$

iii) $p(x)=2x^5-4x^4-6x,$

$q(x)=x^5+x^4-3x^3-3x^2$

$$
\begin{array}{r}
2 \\
x^5+x^4-3x^3-3x^2\big)\overline{\cancel{2x^5}-4x^4-6x} \\
\underline{\cancel{-2x^5}\pm 2x^4\quad\mp 6x^3\mp 6x^2} \\
-6x^4+6x^3+6x^2-6x
\end{array}
$$

Dividing by -6

$x^4-x^3-x^2+x$

$$
\begin{array}{r}
x+2 \\
x^4-x^3-x^2+x\big)\overline{\cancel{x^5}+x^4-3x^3-3x^2} \\
\underline{-\cancel{x^5}\mp x^4\mp x^3\pm x^2} \\
\cancel{2x^4}-\cancel{2x^3}-4x^2 \\
\underline{\cancel{-2x^4}\mp\cancel{2x^3}\mp 2x^2\pm 2x} \\
-2x^2-2x
\end{array}
$$

Dividing by -2

x^2+x

$$
\begin{array}{r}
x^2-2x+1 \\
x^2+x\big)\overline{\cancel{x^4}-x^3-x^2+x} \\
\underline{\cancel{-x^4}\pm x^3} \\
\cancel{-2x^3}-x^2+x \\
\underline{\cancel{\mp 2x^3}\mp 2x^2} \\
\cancel{x^2}+\cancel{x} \\
\underline{\pm\cancel{x^2}\pm\cancel{x}} \\
0
\end{array}
$$

Hence H.C.F $= x^2+x=x(x+1)$

Q4. Find the L.C.M of the following expressions:

i) $39x^7y^3z$ and $91x^5y^6z^7$

Sol: By factorization

$39x^7y^3z=13\times 3\,x.x.x.x.x.x.x.y.y.y.z$

$91x^5y^6z^7=13\times 7\ x.x.x.x.x.y.y.y.y.y.y.z.z.z.z.z.z.z$

Hence L.C.M =

$13\times 3\times 7.\ x.x.x.x.x.x.x.\ y.\ y.y.y.y.y.z.z.z.z.z.z.z$

$=273x^7y^6z^7$

ii) $102xy^2z$, $85x^2yz$ and $187xyz^2$

Sol: By factorization

$102xy^2z = 2\times 3\times 17x.y.y.z$

$85x^2yz = 5\times 17x.x.y.z$

$187xyz^2 = 11\times 17x.y.z.z$

Hence L.C.M $= 17\times 11\times 5\times 3\times 2.x.x.y.y.z.z$

$= 5610x^2y^2z^2$

Q5. Find the L.C.M of the following expressions by factorization:

i) $x^2-25x+100$ and x^2-x-20

Sol: By factorization

$x^2-25x+100 = x^2-5x-20x+100$

$= x(x-5)-20(x-5)$

$= (x-5)(x-20)$.............(i)

$x^2-x-20 = x^2-5x+4x-20$

$= x(x-5)+4(x-5)$

$= (x-5)(x+4)$ ....(ii)

From (i) and (ii)

L.C.M $= (x-5)(x-20)(x+4)$

ii) $x^2+4x+4,\ x^2-4,\ 2x^2+x-6$

Sol: By factorization

$x^2+4x+4 = x^2+2x+2x+4$

$= x(x+2)+2(x+2)$

$= (x+2)(x+2)$ ....(i)

$x^2-4 = (x)^2-(2)^2$

$= (x+2)(x-2)$ ....(ii)

$2x^2+x-6 = 2x^2+4x-3x-6$

$= 2x(x+2)-3(x+2)$

$= (x+2)(2x-3)$ ....(iii)

From (i), (ii) and (iii)

LCM $= (x+2)(x+2)(x-2)(2x-3)$

$= (x+2)^2(x-2)(2x-3)$

iii) $2(x^4-y^4),\ 3(x^3+2x^2y-xy^2-2y^3)$

Sol: By factorization

$2(x^4-y^4) = 2\left[(x^2)^2-(y^2)^2\right]$

$= 2(x^2+y^2)(x^2-y^2)$

$= 2(x^2+y^2)(x+y)(x-y)$ ....(i)

$3(x^3+2x^2y-xy^2-2y^3) = 3\left[x^2(x+2y)-y^2(x+2y)\right]$

$= 3(x+2y)(x^2-y^2)$

$= 3(x+2y)(x+y)(x-y)$ ....(ii)

From (i) & (ii)

L.C.M =

$2\times 3(x+y)(x-y)(x^2+y^2)(x+2y)$

$= 6(x^4-y^4)(x+2y)$

iv) $4(x^4-1), 6(x^3-x^2-x+1)$

Sol: By factorization

$4(x^4-1) = 4\left[(x^2)^2-(1)^2\right]$

$= 4(x^2+1)(x^2-1)$

$= 2\times 2(x^2+1)\left[(x)^2-(1)^2\right]$

$= 2\times 2(x^2+1)(x+1)(x-1)$ ....(i)

$6(x^3-x^2-x+1) = 6\left[x^2(x-1)-1(x-1)\right]$

$= 6(x-1)(x^2-1) = 2\times 3(x-1)\left[(x)^2-(1)^2\right]$

$= 2\times 3(x-1)(x-1)(x+1)$ ....(ii)

From (i) & (ii)

LCM= $2\times 2\times 3(x+1)(x-1)(x^2+1)(x-1)$

$= 12(x^4-1)(x-1)$

Q6. For what value of k is $(x+4)$, the H.C.F of $x^2+x-(2k+2)$ and $2x^2+kx-12$?

Sol: $k = ?$

$p(x) = x^2+x-(2k+2)$ and

$q(x) = 2x^2+kx-12$

As given that $x+4$ is HCF, so $p(x)$ and $q(x)$ will be exactly divisible by $(x+4)$

$$\begin{array}{r} x-3 \\ x+4\overline{)\cancel{x^2}+x-(2k+2)} \\ \underline{\cancel{\pm x^2}\pm 4x} \\ \cancel{-3x}-(2k+2) \\ \underline{\cancel{\mp 3x}\mp 12} \\ 12-(2k+2) \end{array}$$

$$=12-2k-2$$
$$=10-2k$$

As $p(x)$ is exactly divisible by $x+4$, so,

$$10-2k=0$$
$$10=2k$$
$$\frac{10}{2}=k$$
$$k=5$$

Q7. If $(x+3)(x-2)$ is the H.C.F of $p(x)=(x+3)(2x^2-3x+k)$ and $q(x)=(x-2)(3x^2+7x-l)$, find k and l.

Sol: $k=?$ and $l=?$

As $(x+3)(x-2)$ is the H.C.F, so $p(x)$ and $q(x)$ will be exactly divisible by $(x+3)(x-2)$ i.e., $\frac{p(x)}{HCF}$ has remainder zero.

$$\frac{\cancel{(x+3)}(2x^2-3x+k)}{\cancel{(x+3)}(x-2)}=\frac{2x^2-3x+k}{x-2}$$

i.e
$$\begin{array}{r} 2x+1 \\ x-2\overline{)2x^2-3x+k} \\ \underline{\pm 2x^2\mp 4x} \\ x+k \\ \underline{\pm x\mp 2} \\ k+2 \end{array}$$

As remainder = 0, then

$$k+2=0$$
$$\boxed{k=-2}$$

and $\frac{q(x)}{HCF}$ has zero remainder

$$\frac{\cancel{(x-2)}(3x^2+7x-l)}{(x+3)\cancel{(x-2)}}=\frac{3x^2+7x-l}{x+3}$$

$$\begin{array}{r} 3x-2 \\ x+3\overline{)\cancel{3x^2}+7x-l} \\ \underline{\cancel{\pm 3x^2}\pm 9x} \\ \cancel{-2x}-l \\ \underline{\cancel{\mp 2x}\mp 6} \\ -l+6 \end{array}$$

As remainder = 0

$$-l+6=0$$
$$-l=-6$$
$$\Rightarrow \quad \boxed{l=6}$$

Q8. The LCM and HCF of two polynomials $p(x)$ and $q(x)$ are $2(x^4-1)$ and $(x+1)$ (x^2+1) respectively. If $p(x)=x^3+x+1$, find $q(x)$.

Sol: LCM $=2(x^4-1)$,

HCF $=(x+1)(x^2+1)$

$p(x)=x^3+x^2+x+1$, $q(x)=?$

As $p(x)\times q(x)=(LCM)\times(HCF)$

$$q(x)=\frac{(LCM)\times(HCF)}{p(x)}$$

$$=\frac{2(x^4-1)\times(x+1)(x^2+1)}{x^3+x^2+x+1}$$

$$=\frac{2(x^4-1)\cancel{(x^3+x^2+x+1)}}{\cancel{x^3+x^2+x+1}}$$

$$q(x)=2(x^4-1)$$

Q9. Let $p(x)=10(x^2-9)(x^2-3x+2)$ and $q(x)=10x(x+3)(x-1)^2$. If the H.C.F. of $p(x), q(x)$ is $10(x+3)(x-1)$, find their L.C.M.

Sol: $p(x)=10(x^2-9)(x^2-3x+2)$,

$q(x)=10x(x+3)(x-1)^2$

H.C.F. $=10(x+3)(x-1)$, L.C.M $= ?$

As $(L.C.M)\times(H.C.F)=p(x)\times q(x)$

$$\text{L.C.M.} = \frac{p(x)\times q(x)}{H.C.F}$$

$$=\frac{\cancel{10}(x^2-9)(x^2-3x+2)\times 10x(x+3)(x-1)^2}{\cancel{10}(x+3)(x-1)}$$

$$=\frac{(x^2-9)(x^2-3x+2)\times 10x\cancel{(x+3)}\cancel{(x-1)}(x-1)}{\cancel{(x+3)}\cancel{(x-1)}}$$

$=10x(x-1)(x^2-9)(x^2-3x+2)$

$=10x(x-1)(x^2-9)(x^2-x-2x+2)$

$=10x(x-1)(x^2-9)\left[x(x-1)-2(x-1)\right]$

$=10x(x-1)(x^2-9)(x-1)(x-2)$

$=10x(x-1)^2(x^2-9)(x-2)$

Q10. Let the product of L.C.M and H.C.F of two polynomials be $(x+3)^2(x-2)(x+5)$. If one polynomial is $(x+3)(x-2)$ and the second polynomial is $x^2+kx+15$, find the value of k.

Sol: $k=?$

Product of L.C.M. & H.C.F is

$LCM\times HCF=(x+3)^2(x-2)(x+5)$

$p(x)=(x+3)(x-2)$

$q(x)=x^2+kx+15$

As $p(x)\times q(x)=LCM\times HCF$

$(x+3)(x-2)(x^2+kx+15)$

$=(x+3)^2(x-2)(x+5)$

$$x^2+kx+15=\frac{(x+3)\cancel{(x+3)}\cancel{(x-2)}(x+5)}{\cancel{(x+3)}\cancel{(x-2)}}$$

$x^2+kx+15=(x+3)(x+5)$

$x^2+kx+15=x^2+3x+5x+15$

$x^2+kx+15=x^2+8x+15$

Comparing co-efficient of 'x'

$\Rightarrow kx=8x$

$\boxed{k=8}$

Q11. Waqas wishes to distribute 128 bananas and also 176 apples equally among a certain number of children. Find the highest number of the Children. Who can get the fruit in this way?

Sol: No. of bananas = 128

No. of apples = 176

Highest no. of children who get the fruit in this way is H.C.F.

So No. of bananas =

$2\times2\times2\times2\times2\times2\times2$

No. of apples =

$2\times2\times2\times2\times11$

Hence required no. of children =

$2\times2\times2\times2=16$

Example

Simplify

$$\frac{x+3}{x^2-3x+2}+\frac{x+2}{x^2-4x+3}+\frac{x+1}{x^2-5x+6},\ x\neq\ 1,2,3$$

Solution

$$\frac{x+3}{x^2-3x+2}+\frac{x+2}{x^2-4x+3}+\frac{x+1}{x^2-5x+6}$$

$$=\frac{x+3}{x^2-2x-x+2}+\frac{x+2}{x^2-3x-x+3}+\frac{x+1}{x^2-3x-2x+6}$$

$$=\frac{x+3}{x(x-2)-1(x-2)}+\frac{x+2}{x(x-3)-1(x-3)}+\frac{x+1}{x(x-3)-2(x-3)}$$

$$=\frac{x+3}{(x-2)(x-1)}+\frac{x+2}{(x-3)(x-1)}+\frac{x+1}{(x-3)(x-2)}$$

$$=\frac{(x+3)(x-3)+(x+2)(x-2)+(x+1)(x-1)}{(x-1)(x-2)(x-3)}$$

$$=\frac{x^2-9+x^2-4+x^2-1}{(x-1)(x-2)(x-3)}$$

$$=\frac{3x^2-14}{(x-1)(x-2)(x-3)}$$

Example

Express the product $\frac{x^3-8}{x^2-4}\times\frac{x^2+6x+8}{x^2-2x+1}$ as an algebraic expression reduced lowest forms $x\neq 2,-2,1$

Solution

By factorizing completely, we have

$$\frac{x^3-8}{x^2-4}\times\frac{x^2+6x+8}{x^2-2x+1}$$

$$=\frac{(x-2)(x^2+2x+4)\times(x+2)(x+4)}{(x-2)(x+2)\times(x-1)^2}\quad ..(i)$$

Now the factors of numerator are $(x-2),(x^2+2x+4),(x+2)$ and $(x+4)$ and the factors of denominator are

$(x-2),(x+2)$ and $(x-1)^2$.

Therefore, their H.C.F. is (x – 2) × (x + 2)

By cancelling H.C.F i.e., $(x-2)\times(x+2)$ from (i), we get the simplified form of given product as the fraction $\frac{(x^2+2x+4)(x+4)}{(x-1)^2}$

Example

Divide $\frac{x^2+x+1}{x^2-9}$ by $\frac{x^3-1}{x^2-4x+3}$ and simplify by reducing to lowest forms.

Solution

We have $\frac{x^2+x+1}{x^2-9}\div\frac{x^3-1}{x^2-4x+3}$

$$=\frac{(x^2+x+1)}{(x^2-9)}\times\frac{(x^2-4x+3)}{(x^3-1)}$$

$$=\frac{(x^2+x+1)(x^2-x-3x+3)}{(x^2-9)(x^3-1)}$$

$$=\frac{\left(x^2+x+1\right)\left[x(x-1)-3(x-1)\right]}{(x+3)(x-3)(x-1)\left(x^2+x+1\right)}$$

$$=\frac{(x^2+x+1)(x-3)(x-1)}{(x+3)(x-3)(x-1)(x^2+x+1)}=\frac{1}{x+3},\ x\neq-3$$

Exercise 6.2

Simplify each of the following as a rational expression.

Q1. $\frac{x^2-x-6}{x^2-9}+\frac{x^2+2x-24}{x^2-x-12}$

$$=\frac{x^2-3x+2x-6}{(x)^2-(3)^2}+\frac{x^2+6x-4x-24}{x^2+3x-4x-12}$$

$$=\frac{x(x-3)+2(x-3)}{(x+3)(x-3)}+\frac{x(x+6)-4(x+6)}{x(x+3)-4(x+3)}$$

$$=\frac{(x-3)(x+2)}{(x+3)(x-3)}+\frac{(x+6)(x-4)}{(x+3)(x-4)}$$

$$=\frac{x+2}{x+3}+\frac{x+6}{x+3}=\frac{x+2+x+6}{x+3}$$

$$=\frac{2x+8}{x+3}$$

$$=\frac{2(x+4)}{x+3}$$

Q2. $$\left[\frac{x+1}{x-1}-\frac{x-1}{x+1}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}$$

$$=\left[\frac{(x+1)^2-(x-1)^2}{(x-1)(x+1)}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}$$

$$=\left[\frac{(x^2+2x+1)-(x^2-2x+1)}{(x)^2-(1)^2}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}$$

$$=\left[\frac{x^2+2x+1-x^2+2x-1}{x^2-1}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}$$

$$=\left[\frac{4x}{x^2-1}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}$$

$$=\left[\frac{4x(x^2+1)-4x(x^2-1)}{(x^2-1)(x^2+1)}\right]+\frac{4x}{x^4-1}$$

$$=\frac{4x^3+4x-4x^3+4x}{(x^2)^2-(1)^2}+\frac{4x}{x^4-1}$$

$$=\frac{8x}{x^4-1}+\frac{4x}{x^4-1}$$

$$=\frac{8x+4x}{x^4-1}$$

$$=\frac{12x}{x^4-1}$$

Q3. $$\frac{1}{x^2-8x+15}+\frac{1}{x^2-4x+3}-\frac{2}{x^2-6x+5}$$

$$=\frac{1}{x^2-3x-5x+15}+\frac{1}{x^2-3x-x+3}-\frac{2}{x^2-5x-x+5}$$

$$=\frac{1}{x(x-3)-5(x-3)}+\frac{1}{x(x-3)-1(x-3)}-\frac{2}{x(x-5)-1(x-5)}$$

$$=\frac{1}{(x-3)(x-5)}+\frac{1}{(x-3)(x-1)}-\frac{2}{(x-5)(x-1)}$$

$$=\frac{x-1+x-5-2(x-3)}{(x-1)(x-3)(x-5)}$$

$$=\frac{x-1+x-5-2x+6}{(x-1)(x-3)(x-5)}$$

$$=\frac{2x-6-2x+6}{(x-1)(x-3)(x-5)}$$

$$=\frac{0}{(x-1)(x-3)(x-5)}$$

$$=0$$

Q4. $$\frac{(x+2)(x+3)}{x^2-9}+\frac{(x+2)(2x^2-32)}{(x-4)(x^2-x-6)}$$

$$=\frac{(x+2)(x+3)}{(x)^2-(3)^2}+\frac{(x+2).2(x^2-16)}{(x-4)(x^2+2x-3x-6)}$$

$$=\frac{(x+2)(x+3)}{(x-3)(x+3)}+\frac{2(x+2)\left[(x)^2-(4)^2\right]}{(x-4)(x^2+2x-3x-6)}$$

$$=\frac{(x+2)}{x-3}+\frac{2(x+2)(x+4)(x-4)}{(x-4)(x+2)(x-3)}$$

$$=\frac{x+2}{x-3}+\frac{2x+8}{x-3}$$

$$=\frac{x+2+2x+8}{x-3}$$

$$=\frac{3x+10}{x-3}$$

Q5. $$\frac{x+3}{2x^2+9x+9}+\frac{1}{2(2x-3)}-\frac{4x}{4x^2-9}$$

$$=\frac{x+3}{2x^2+6x+3x+9}+\frac{1}{2(2x-3)}-\frac{4x}{(2x)^2-(3)^2}$$

$$=\frac{x+3}{2x(x+3)+3(x+3)}+\frac{1}{2(2x-3)}-\frac{4x}{(2x+3)(2x-3)}$$

$$=\frac{\cancel{x+3}}{\cancel{(x+3)}(2x+3)}+\frac{1}{2(2x-3)}-\frac{4x}{(2x+3)(2x-3)}$$

$$=\frac{1}{2x+3}+\frac{1}{2(2x-3)}-\frac{4x}{(2x+3)(2x-3)}$$

$$=\frac{2(2x-3)+2x+3-2(4x)}{2(2x+3)(2x-3)}$$

$$=\frac{4x-6+2x+3-8x}{2(2x+3)(2x-3)}$$

$$=\frac{-2x-3}{2(2x+3)(2x-3)}$$

$$=\frac{-1\cancel{(2x+3)}}{2\cancel{(2x+3)}(2x-3)}$$

$$=\frac{-1}{2(2x-3)}$$

$$=\frac{1}{2(3-2x)}$$

Q6. $A-\frac{1}{A}$, **where** $A=\frac{a+1}{a-1}$

so $\frac{1}{A}=\frac{a-1}{a+1}$

Now $A-\frac{1}{A}=\frac{a+1}{a-1}-\frac{a-1}{a+1}$

$$=\frac{(a+1)^2-(a-1)^2}{(a-1)(a+1)}$$

$$=\frac{(a^2+2a+1)-(a^2-2a+1)}{(a)^2-(1)^2}$$

$$=\frac{\cancel{a^2}+2a+\cancel{1}-\cancel{a^2}+2a-\cancel{1}}{a^2-1}$$

$$=\frac{4a}{a^2-1}$$

Q7. $\left[\frac{x-1}{x-2}+\frac{2}{2-x}\right]-\left[\frac{x+1}{x+2}+\frac{4}{4-x^2}\right]$

$$=\left[\frac{-(x-1)}{2-x}+\frac{2}{2-x}\right]-\left[\frac{x+1}{x+2}+\frac{4}{(2)^2-(x)^2}\right]$$

$$=\left[-\frac{(x-1)}{2-x}+\frac{2}{2-x}\right]-\left[\frac{x+1}{x+2}+\frac{4}{(2+x)(2-x)}\right]$$

$$=\left[\frac{-x+1+2}{2-x}\right]-\left[\frac{(x+1)(2-x)+4}{(2+x)(2-x)}\right]$$

$$=\frac{3-x}{2-x}-\left[\frac{2x-x^2+2-x+4}{(2+x)(2-x)}\right]$$

$$=\frac{3-x}{2-x}-\left[\frac{6+x-x^2}{(2+x)(2-x)}\right]$$

$$=\frac{3-x}{2-x}-\left[\frac{6+3x-2x-x^2}{(2+x)(2-x)}\right]$$

$$=\frac{3-x}{2-x}-\left[\frac{3(2+x)-x(2+x)}{(2+x)(2-x)}\right]$$

$$=\frac{3-x}{2-x}-\left[\frac{\cancel{(2+x)}(3-x)}{\cancel{(2+x)}(2-x)}\right]$$

$$=\frac{3-x}{2-x}-\frac{3-x}{2-x}$$

$$=\frac{3-x-3+x}{2-x}$$

$$=\frac{0}{2-x}$$

$$=0$$

Q8. What rational expression should be subtracted from $\frac{2x^2+2x-7}{x^2+x-6}$ **to get** $\frac{x-1}{x-2}=?$

Sol: Let the required expression be A,

then $\dfrac{2x^2+2x-7}{x^2+x-6}-A=\dfrac{x-1}{x-2}$

or $\dfrac{2x^2+2x-7}{x^2+x-6}-\dfrac{x-1}{x-2}=A$

So $A=\dfrac{2x^2+2x-7}{x^2+3x-2x-6}-\dfrac{x-1}{x-2}$

$$=\frac{2x^2+2x-7}{x(x+3)-2(x+3)}-\frac{x-1}{x-2}$$

$$=\frac{2x^2+2x-7}{(x+3)(x-2)}-\frac{x-1}{x-2}$$

$$=\frac{2x^2+2x-7-(x-1)(x+3)}{(x+3)(x-2)}$$

$$=\frac{2x^2+2x-7-(x^2-x+3x-3)}{(x+3)(x-2)}$$

$$=\frac{(2x^2+2x-7)-(x^2+2x-3)}{(x+3)(x-2)}$$

$$=\frac{2x^2+2x-7-x^2-2x+3}{(x+3)(x-2)}$$

$$=\frac{x^2-4}{(x+3)(x-2)}$$

$$=\frac{(x)^2-(2)^2}{(x+3)(x-2)}$$

$$=\frac{(x+2)\cancel{(x-2)}}{(x+3)\cancel{(x-2)}}$$

$$=\frac{x+2}{x+3}$$

Perform the indicated operations and simplify to the lowest forms.

Q9. $\dfrac{x^2+x-6}{x^2-x-6}\times\dfrac{x^2-4}{x^2-9}$

$$=\frac{x^2+3x-2x-6}{x^2-3x+2x-6}\times\frac{(x)^2-(2)^2}{(x)^2-(3)^2}$$

$$=\frac{x(x+3)-2(x+3)}{x(x-3)+2(x-3)}\times\frac{(x+2)(x-2)}{(x+3)(x-3)}$$

$$=\frac{\cancel{(x+3)}(x-2)}{(x-3)\cancel{(x+2)}}\times\frac{\cancel{(x+2)}(x-2)}{\cancel{(x+3)}(x-3)}$$

$$=\frac{(x-2)^2}{(x-3)^2}$$

Q10. $\dfrac{x^3-8}{x^2-4}\times\dfrac{x^2+6x+8}{x^2-2x+1}$

$$=\frac{(x)^3-(2)^3}{(x)^2-(2)^2}\times\frac{x^2+2x+4x+8}{x^2-x-x+1}$$

$$=\frac{\cancel{(x-2)}\left[(x)^2+(x)(2)+(2)^2\right]}{\cancel{(x-2)}(x+2)}\times\frac{x(x+2)+4(x+2)}{x(x-1)-1(x-1)}$$

$$=\frac{x^2+2x+4}{\cancel{x+2}}\times\frac{\cancel{(x+2)}(x+4)}{(x-1)(x-1)}$$

$$=\frac{(x^2+2x+4)(x+4)}{(x-1)^2}$$

Q11. $\dfrac{x^4-8x}{2x^2+5x-3}\times\dfrac{2x-1}{x^2+2x+4}\times\dfrac{x+3}{x^2-2x}$

$$=\frac{x(x^3-8)}{2x^2+6x-x-3}\times\frac{2x-1}{x^2+2x+4}\times\frac{x+3}{x(x-2)}$$

$$=\frac{x\left[(x)^3-(2)^3\right]}{2x(x+3)-1(x+3)}\times\frac{2x-1}{x^2+2x+4}\times\frac{x+3}{x(x-2)}$$

$$=\frac{\cancel{x}\cancel{(x-2)}\cancel{(x^2+2x+4)}}{\cancel{(x+3)}\cancel{(2x-1)}}\times\frac{\cancel{2x-1}}{\cancel{x^2+2x+4}}\times\frac{\cancel{x+3}}{\cancel{x}\cancel{(x-2)}}$$

$$=1$$

Q12. $\dfrac{2y^2+7y-4}{3y^2-13y+4}\div\dfrac{4y^2-1}{6y^2+y-1}$

$$=\frac{2y^2+8y-y-4}{3y^2-y-12y+4}\div\frac{(2y)^2-(1)^2}{6y^2+3y-2y-1}$$

$$=\frac{2y(y+4)-1(y+4)}{y(3y-1)-4(3y-1)}\div\frac{(2y+1)(2y-1)}{3y(2y+1)-1(2y+1)}$$

$$=\frac{(y+4)\cancel{(2y-1)}}{\cancel{(3y-1)}(y-4)}\div\frac{\cancel{(2y+1)}(2y-1)}{\cancel{(2y+1)}\cancel{(3y-1)}}$$

$$=\frac{(y+4)\cancel{(2y-1)}}{\cancel{(3y-1)}(y-4)}\times\frac{\cancel{(2y+1)}(3y-1)}{\cancel{(2y+1)}\cancel{(2y-1)}}$$

$$=\frac{y+4}{y-4}$$

Q13. $\left[\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right]\div\left[\frac{x+y}{x-y}-\frac{x-y}{x+y}\right]$

$$=\left[\frac{\left(x^2+y^2\right)^2-\left(x^2-y^2\right)^2}{\left(x^2-y^2\right)\left(x^2+y^2\right)}\right]\div\left[\frac{(x+y)^2-(x-y)^2}{(x-y)(x+y)}\right]$$

$$=\frac{x^4+y^4+2x^2y^2-\left(x^4+y^4-2x^2y^2\right)^2}{\left(x^2-y^2\right)\left(x^2+y^2\right)}$$

$$\div\frac{x^2+y^2+2xy-x^2-y^2+2xy}{x^2-y^2}$$

$$=\frac{\cancel{x^4}+\cancel{y^4}+2x^2y^2-\cancel{x^4}-\cancel{y^4}+2x^2y^2}{\left(x^2-y^2\right)\left(x^2+y^2\right)}$$

$$\div\frac{\cancel{x^2}+\cancel{y^2}+2xy-\cancel{x^2}-\cancel{y^2}+2xy}{x^2-y^2}$$

$$=\frac{4x^2y^2}{\left(x^2-y^2\right)\left(x^2+y^2\right)}\div\frac{4xy}{x^2-y^2}$$

$$=\frac{\cancel{4}\cancel{x^2}\cancel{y^2}}{\cancel{\left(x^2-y^2\right)}\left(x^2+y^2\right)}\times\frac{\cancel{x^2-y^2}}{\cancel{4xy}}$$

$$=\frac{xy}{x^2+y^2}$$

Square Root of Algebraic Expression

The square root of a given expression $p(x)$ as another expression $q(x)$ such that $q(x).q(x) = p(x)$.

As $5\times5=25$, so square root of 25 is 5

It means we can find square root of the expression $p(x)$ if it can be expressed as a perfect square.

Example

Use factorization to find the square root of the expression

$$4x^2-12x+9$$

Solution

We have, $4x^2-12x+9$

$$=4x^2-6x-6x+9=2x(2x-3)-3(2x-3)$$

$$=\quad(2x-3)(2x-3)=(2x-3)^2$$

Hence $\sqrt{4x^2-12x+9}$

$$=\quad\pm(2x-3)$$

Example

Find the square root of

$$x^2+\frac{1}{x^2}+12\left(x+\frac{1}{x}\right)+38, x\neq0$$

Solution

We have $x^2+\frac{1}{x^2}+12\left(x+\frac{1}{x}\right)+38$

$$=\quad x^2+\frac{1}{x^2}+2+12\left(x+\frac{1}{x}\right)+36,$$

(adding and subtracting 2)

$$= \left(x+\frac{1}{x}\right)^2+2\left(x+\frac{1}{x}\right)(6)+(6)^2$$

$$= \left[\pm\left(x+\frac{1}{x}+6\right)\right]^2;$$

since $a^2+2ab+b^2=(a+b)^2$

Hence the required square root is $\pm\left(x+\frac{1}{x}+6\right)$

Example

Find the square root of $4x^4+12x^3+x^2-12x+4$

Solution

$$\begin{array}{r|l}
 & 2x^2+3x-2 \\
\hline
2x^2 & \cancel{4x^4}+12x^3+x^2-12x+4 \\
 & \underset{-}{\cancel{4x^4}} \\
\hline
4x^2+3x & \cancel{12x^3}+x^2-12x+4 \\
 & \underset{-}{\cancel{12x^3}}\ \pm 9x^2 \\
\hline
4x^2+6x-2 & -8x^2-12x+4 \\
 & \mp 8x^2 \mp 12x \pm 4 \\
\cline{2-2}
 & 0
\end{array}$$

Thus square root of given expression is $\pm(2x^2+3x-2)$

Example 2

Find the square root of the expression

$$4\frac{x^2}{y^2}+8\frac{x}{y}+16+12\frac{y}{x}+9\frac{y^2}{x^2}$$

Solution

We note that the given expression is in descending powers of x.

$$\begin{array}{r|l}
 & 2\frac{x}{y}+2+3\frac{y}{x} \\
\hline
2\frac{x}{y} & 4\frac{x^2}{y^2}+8\frac{x}{y}+16+12\frac{y}{x}+9\frac{y^2}{x^2} \\
 & \cancel{\pm 4\frac{x^2}{y^2}} \\
\hline
4\frac{x}{y}+2 & 8\frac{x}{y}\ +16 \\
 & \pm 8\frac{x}{y} \pm 4 \\
\hline
4\frac{x}{y}+4+3\frac{y}{x} & 12+12\frac{y}{x}+9\frac{y^2}{x^2} \\
 & \pm 12 \pm 12\frac{y}{x} \pm 9\frac{y^2}{x^2} \\
\cline{2-2}
 & 0
\end{array}$$

Hence the square root of given expression is $\pm\left(2\frac{x}{y}+2+3\frac{y}{x}\right)$

Example

To make the expression $x^4-10x^3+33x^2-42x+20$ a perfect square,

(i) What should be added to it?

(ii) What should be subtracted from it?

(iii) What should be the value of x?

$$\begin{array}{r|l}
 & x^2-5x+4 \\
\hline
x^2 & x^4-10x^3+33x^2-42x+20 \\
 & \pm x^4 \\
\hline
2x^2-5x & -10x^3+33x^2 \\
 & \underset{+}{-}10x^3\underset{-}{+}25x^2 \\
\hline
2x^2-10x+4 & 8x^2-42x+20 \\
 & -8x^2\underset{+}{-}40x\underset{-}{+}16 \\
\cline{2-2}
 & -2x+4 \\
\cline{2-2}
\end{array}$$

For making the given expression a perfect square the remainder must be zero.

Hence

(i) We should add $(2x-4)$ to the given expression

(ii) We should subtract $(-2x+4)$ from the given expression

(iii) We should take $-2x+4=0$ to find the value of x. This gives the required value of x i.e., $x=2$.

Exercise 6.3

Q1. Use factorization to find the square root of the following expressions.

i) $4x^2-12xy+9y^2$

$=(2x)^2-2(2x)(3y)+(3y)^2$

$=(2x-3y)^2$

Hence $\sqrt{4x^2-12xy+9y^2}$

$=\sqrt{(2x-3y)^2}$

$=\pm(2x-3y)$

ii) $x^2-1+\frac{1}{4x^2}$

$=(x)^2-2(x)\left(\frac{1}{2x}\right)+\left(\frac{1}{2x}\right)^2$

Hence $\sqrt{x^2-1+\frac{1}{4x^2}}$

$=\sqrt{\left(x-\frac{1}{2x}\right)^2}$

$=\pm\left(x-\frac{1}{2x}\right)$

iii) $\frac{1}{16}x^2-\frac{1}{12}xy+\frac{1}{36}y^2$

$=\left(\frac{1}{4}x\right)^2-2\left(\frac{1}{4}x\right)\left(\frac{1}{6}y\right)+\left(\frac{1}{6}y\right)^2$

$=\left(\frac{1}{4}x-\frac{1}{6}y\right)^2$

Hence $\sqrt{\frac{1}{16}x^2-\frac{1}{12}xy+\frac{1}{36}y^2}$

$=\sqrt{\left(\frac{1}{4}x-\frac{1}{6}y\right)^2}$

$=\pm\left(\frac{1}{4}x-\frac{1}{6}y\right)$

iv) $4(a+b)^2-12(a^2-b^2)+9(a-b)^2$

$=[2(a+b)]^2-2\times2(a+b)\times3(a-b)+[3(a-b)]^2$

$=[2(a+b)-3(a-b)]^2$

$=(-a+5b)^2$

$=(5b-a)^2$

Hence $\sqrt{4(a+b)^2-12(a^2-b^2)+9(a-b)^2}$

$=\sqrt{(5b-a)^2}$

$=\pm(5b-a)$

v) $\frac{4x^6-12x^3y^3+9y^6}{9x^4+24x^2y^2+16y^4}$

$=\frac{(2x^3)^2-2(2x^3)(3y^3)+(3y^3)^2}{(3x^2)^2+2(3x^2)(4y^2)+(4y^2)^2}$

$$=\frac{\left(2x^3-3y^3\right)^2}{\left(3x^2+4y^2\right)^2}$$

Hence $\sqrt{\dfrac{4x^6-12x^3y^3+9y^6}{9x^4+24x^2y^2+16y^4}}$

$$=\sqrt{\left(\frac{2x^3-3y^3}{3x^2+4y^2}\right)^2}$$

$$=\pm\left(\frac{2x^3-3y^3}{3x^2+4y^2}\right).$$

vi) $\left(x+\dfrac{1}{x}\right)^2-4\left(x-\dfrac{1}{x}\right) \qquad (x\neq 0)$

$$=(x)^2+\left(\frac{1}{x}\right)^2+2(\cancel{x})\left(\frac{1}{\cancel{x}}\right)-4\left(x-\frac{1}{x}\right)$$

$$=x^2+\frac{1}{x^2}+2-4\left(x-\frac{1}{x}\right)\ldots\ldots\text{(i)}$$

Let $x-\dfrac{1}{x}=a$

Squaring $\left(x-\dfrac{1}{x}\right)^2=(a)^2$

$$x^2+\frac{1}{x^2}-2=a^2$$

$$x^2+\frac{1}{x^2}=a^2+2$$

So expression (i) becomes

$$=a^2+2+2-4a$$

$$=a^2-4a+4$$

$$=(a)^2-2(a)(2)+(2)^2$$

$$=(a-2)^2$$

Putting value of 'a'

$$=\left(x-\frac{1}{x}-2\right)^2$$

Hence $=\sqrt{\left(x-\dfrac{1}{x}-2\right)^2}$

$$=\pm\left(x-\frac{1}{x}-2\right)$$

vii) $\left(x^2+\dfrac{1}{x^2}\right)^2-4\left(x+\dfrac{1}{x}\right)^2+12\ldots\text{(i)}$

Let $x+\dfrac{1}{x}=a$

Squaring $\left(x+\dfrac{1}{x}\right)^2=(a)^2$

$$x^2+\frac{1}{x^2}+2=a^2$$

$$x^2+\frac{1}{x^2}=a^2-2$$

So expression (i) becomes

$$=\left(a^2-2\right)^2-4(a)^2+12$$

$$=\left(a^2\right)^2-2\left(a^2\right)(2)+(2)^2-4a^2+12$$

$$=a^4-4a^2+4-4a^2+12$$

$$=a^4-8a^2+16$$

$$=\left(a^2\right)^2-2\left(a^2\right)(4)+(4)^2$$

$$=\left(a^2-4\right)^2$$

Putting values of a^2

$$=\left(x^2+\frac{1}{x^2}+2-4\right)^2$$

$$=\left(x^2+\frac{1}{x^2}-2\right)^2$$

Hence $=\sqrt{\left(x^2+\dfrac{1}{x^2}\right)^2-4\left(x+\dfrac{1}{x}\right)^2+12}$

$$=\sqrt{\left(x^2+\frac{1}{x^2}-2\right)^2}$$

$$=\pm\left(x^2+\frac{1}{x^2}-2\right)$$

viii) $\left(x^2+3x+2\right)\left(x^2+4x+3\right)\left(x^2+5x+6\right)$

$$=\left(x^2+x+2x+2\right)\left(x^2+x+3x+3\right)\left(x^2+2x+3x+6\right)$$

$$=\left[x(x+1)+2(x+1)\right]\left[x(x+1)+3(x+1)\right]\left[x(x+2)+3(x+2)\right]$$

$$=(x+1)(x+2)(x+1)(x+3)(x+2)(x+3)$$

$$=(x+1)^2(x+2)^2(x+3)^2$$

Hence

$$\sqrt{\left(x^2+3x+2\right)\left(x^2+4x+3\right)\left(x^2+5x+6\right)}$$

$$=\sqrt{(x+1)^2(x+2)^2(x+3)^2}$$

$$=\pm(x+1)(x+2)(x+3)$$

ix) $\left(x^2+8x+7\right)\left(2x^2-x-3\right)\left(2x^2+11x-21\right)$

$$=\left(x^2+x+7x+7\right)\left(2x^2+2x-3x-3\right)\left(2x^2+14x-3x-21\right)$$

$$=\left[x(x+1)+7(x+1)\right]\left[2x(x+1)-3(x+1)\right]$$

$$\left[2x(x+7)-3(x+7)\right]$$

$$=(x+1)(x+7)(x+1)(2x-3)(x+7)(2x-3)$$

$$=(x+1)^2(x+7)^2(2x-3)^2$$

Hence

$$\sqrt{\left(x^2+8x+7\right)\left(2x^2-x-3\right)\left(2x^2+11x-21\right)}$$

$$=\sqrt{(x+1)^2(x+7)^2(2x-3)^2}$$

$$=\pm(x+1)(x+7)(2x-3)$$

Q2. Use division method to find the square root of the following expressions.

i) $4x^2+12xy+9y^2+16x+24y+16$

$$\begin{array}{r|l} & 2x+3y+4 \\ \hline 2x & 4x^2+12xy+9y^2+16x+24y+16 \\ & \underline{4x^2} \\ \hline 4x+3y & \cancel{12xy}+\cancel{9y^2}+16x+24y+16 \\ & \cancel{12xy}\pm\cancel{9y^2} \\ \hline 4x+6y+4 & \cancel{16x}+\cancel{24y}+\cancel{16} \\ & \cancel{16x}\pm\cancel{24y}\pm\cancel{16} \\ \hline & 0 \end{array}$$

Hence the square root of given expression is

$\pm(2x+3y+4)$

ii) $x^4-10x^3+37x^2-60x+36$

$$\begin{array}{r|l} & x^2-5x+6 \\ \hline x^2 & x^4-10x^3+37x^2-60x-36 \\ & -x^4 \\ \hline 2x^2-5x & \cancel{-10x^3}+37x^2-60x-36 \\ & \cancel{\mp10x^3}\pm25x^2 \\ \hline 2x^2-10x+6 & \cancel{-12x^2}-\cancel{60x}+\cancel{36} \\ & \cancel{-12x^2}\mp\cancel{60x}\pm\cancel{36} \\ \hline & 0 \end{array}$$

Hence $\sqrt{x^4-10x^3+37x^2-60x+36}$

$$=\pm\left(x^2-5x+6\right)$$

iii) $9x^4-6x^3+7x^2-2x+1$

$$\begin{array}{r|l}
 & 3x^2 - x + 1 \\ \hline
3x^2 & 9x^4 - 6x^3 + 7x^2 - 2x + 1 \\
 & -9x^4 \\ \hline
6x^2 - x & \cancel{-6x^3} + 7x^2 - 2x + 1 \\
 & \cancel{\mp 6x^3} \pm x^2 \\ \hline
6x^2 - 2x + 1 & \cancel{6x^2} - \cancel{2x} + \cancel{1} \\
 & \cancel{-6x^2} \mp \cancel{2x} \pm \cancel{1} \\ \hline
 & 0
\end{array}$$

Hence $\sqrt{9x^4 - 6x^3 + 7x^2 - 2x + 1}$

$= \pm\left(3x^2 - x + 1\right)$

iv) $4 + 25x^2 - 12x - 24x^3 + 16x^4$

In descending order

$= 16x^4 - 24x^3 + 25x^2 - 12x + 4$

$$\begin{array}{r|l}
 & 4x^2 - 3x + 2 \\ \hline
4x^2 & 16x^4 - 24x^3 + 25x^2 - 12x + 4 \\
 & -16x^4 \\ \hline
8x^2 - 3x & \cancel{-24x^3} + 25x^2 - 12x + 4 \\
 & \cancel{\mp 24x^3} \pm 9x^2 \\ \hline
8x^2 - 6x + 2 & \cancel{16x^2} - \cancel{12x} + \cancel{4} \\
 & \cancel{-16x^2} \mp \cancel{12x} \pm \cancel{4} \\ \hline
 & 0
\end{array}$$

Hence $\sqrt{16x^4 - 24x^3 + 25x^2 - 12x + 4}$

$= \pm\left(4x^2 - 3x + 2\right)$

v) $\dfrac{x^2}{y^2} - 10\dfrac{x}{y} + 27 - 10\dfrac{y}{x} + \dfrac{y^2}{x^2}$

$(x \neq 0,\ y \neq 0)$

Hence

$$\begin{array}{r|l}
 & \frac{x}{y} - 5 + \frac{y}{x} \\ \hline
\frac{x}{y} & \frac{x^2}{y^2} - 10\frac{x}{y} + 27 - 10\frac{y}{x} + \frac{y^2}{x^2} \\
 & -\frac{x^2}{y^2} \\ \hline
\frac{2x}{y} - 5 & \cancel{-10\frac{x}{y}} + 27 - 10\frac{y}{x} + \frac{y^2}{x^2} \\
 & \cancel{\mp 10\frac{x}{y}} \pm 25 \\ \hline
\frac{2x}{y} - 10 + \frac{y}{x} & \cancel{2} - \cancel{10\frac{y}{x}} + \cancel{\frac{y^2}{x^2}} \\
 & \cancel{-2} \mp \cancel{10\frac{y}{x}} \pm \cancel{\frac{y^2}{x^2}} \\ \hline
 & 0
\end{array}$$

$\sqrt{\dfrac{x^2}{y^2} - 10\dfrac{x}{y} + 27 - 10\dfrac{y}{x} + \dfrac{y^2}{x^2}}$

The required square root

$= \pm\left(\dfrac{x}{y} - 5 + \dfrac{y}{x}\right)$

Q3. **Find the value of 'k' for which the following expression will become a perfect square?**

i) $4x^4 - 12x^3 + 37x^2 - 42x + k$

$$\begin{array}{r|l}
 & 2x^2 - 3x + 7 \\ \hline
2x^2 & 4x^4 - 12x^3 + 37x^2 - 42x + k \\
 & -4x^4 \\ \hline
4x^2 - 3x & \cancel{-12x^3} + 37x^2 - 42x + k \\
 & \cancel{\mp 12x^3} \pm 9x^2 \\ \hline
4x^2 - 6x + 7 & \cancel{28x^2} - \cancel{42x} + k \\
 & \cancel{-28x^2} \mp \cancel{42x} \pm 49 \\ \hline
 & k - 49
\end{array}$$

As given that the given expression is a perfect square, so

$$\text{Remainder} = 0$$
$$k - 49 = 0$$

$$\boxed{k=49}$$

ii) $x^4-4x^3+10x^2-kx+9$

$$
\begin{array}{r|l}
 & x^2-2x+3 \\ \hline
x^2 & x^4-4x^3+10x^2-kx+9 \\
 & -x^4 \\ \hline
2x^2-2x & -\cancel{4x^3}+10x^2-kx+9 \\
 & \mp\cancel{4x^3}\pm 4x^2 \\ \hline
2x^2-4x+3 & \cancel{6x^2}-kx+\cancel{9} \\
 & -\cancel{6x^2}\mp 12x\pm\cancel{9} \\ \hline
 & (-k+12)x
\end{array}
$$

As given that the given expression is a perfect square, so

$$\text{Remainder}=0$$
$$(-k+12)x=0$$

As $x\neq 0$, so $-k+12=0$

$$\Rightarrow \boxed{k=12}$$

Q4. Find the values of 'l' and 'm' for which the following expression will become perfect square.

i) $x^4+4x^3+16x^2+lx+m$

$$
\begin{array}{r|l}
 & x^2+2x+6 \\ \hline
x^2 & \cancel{x^4}+4x^3+16x^2+lx+m \\
 & -\cancel{x^4} \\ \hline
2x^2+2x & \cancel{4x^3}+16x^2+lx+m \\
 & -\cancel{4x^3}\pm 4x^2 \\ \hline
2x^2+4x+6 & \cancel{12x^2}+lx+m \\
 & -\cancel{12x^2}\pm 24x\pm 36 \\ \hline
 & (l-24)x+(m-36)
\end{array}
$$

As the given expression is to be a perfect square, so

$$\text{Remainder}=0$$
$$(l-24)x+(m-36)=0$$

As $x\neq 0$, so $l-24=0$ and $m-36=0$

$\Rightarrow \boxed{l=24}$ and $\boxed{m=36}$

ii) $49x^4-70x^3+109x^2+lx-m$

$$
\begin{array}{r|l}
 & 7x^2-5x+6 \\ \hline
7x^2 & 49x^4-70x^3+109x^2+lx-m \\
 & -49x^4 \\ \hline
14x^2-5x & -\cancel{70x^3}+109x^2+lx-m \\
 & \mp\cancel{70x^3}\pm 25x^2 \\ \hline
14x^2-10x+6 & \cancel{84x^2}+lx-m \\
 & -\cancel{84x^2}\mp 60x\pm 36 \\ \hline
 & (l+60)x-m-36
\end{array}
$$

As the given expression is to be a perfect square, so

$$(l+60)x-m-36=0$$

As $x\neq 0$, so $l+60=0$ and $-m-36=0$

$\Rightarrow \boxed{l=-60}$ and $\boxed{m=-36}$

Q5. To make the expression $9x^4-12x^3+22x^2-13x+12$ a perfect square.

i) What should be added to it?

ii) What should be subtracted from it?

iii) What should be the value of 'x'?

$$
\begin{array}{r|l}
 & 3x^2-2x+3 \\ \hline
3x^2 & 9x^4-12x^3+22x^2-13x+12 \\
 & -9x^4 \\ \hline
6x^2-2x & -\cancel{12x^3}+22x^2-13x+12 \\
 & \mp\cancel{12x^3}\pm 4x^2 \\ \hline
6x^2-4x+3 & \cancel{18x^2}-13x+12 \\
 & -\cancel{18x^2}\mp 12x\pm 9 \\ \hline
 & -x+3
\end{array}
$$

To make the given expression a complete square

i) $x-3$ should be added

ii) $-x+3$ should be subtracted

iii) For value of 'x'

Remainder = 0

$$-x+3=0$$

$$\boxed{x=3}$$

Q6. Find H.C.F of following by factorization

$8x^4 - 128, 12x^3 - 96$.

Solution:

$8x^4 - 128 = 8(x^4 - 16)$

$= 8((x^2)^2 - (4)^2)$

$= 8(x^2 + 4)(x^2 - 4)$

$= 8(x^2+4)(x+2)(x-2)$

$12x^3 - 96 = 12(x^3 - 8)$

$= 12(x^3 - 2^3)$

$= 12(x-2)(x^2+2x+4)$

Common factor $= 4(x-2)$

H.C.F $= 4(x-2)$

Q7. Find H.C.F of following by division method.

$y^3 + 3y^2 - 3y - 9, y^3 + 3y^2 - 8y - 24$

Solution:

$$1$$

$$y^3+3y^2-3y-9 \quad y^3+3y^2-8y-24$$

$$-y^3 \pm 3y^2 \mp 3y \mp 9$$

$$-5y-15$$

$$-5(y+3)$$

$$y^2-3$$

$$(y+3) \quad y^3+3y^2-3y-9$$

$$-y^3 \pm 3y^2$$

$$-3y-9$$

$$\mp 3y \pm 9$$

$$\times$$

H.C.F = y + 3

Q8. Find L.C.M of following by factorization.

$12x^2\ 75, 6x^2 - 13x - 5, 4x^2 - 20x + 25$

Solution:

$12x^2 - 75 = 3(4x^2 - 25)$

$= 3((2x)^2 - (5)^2)$

$= 3(2x+5)(2x-5)$

$6x^2 - 13x - 5 = 6x^2 - 15x + 2x - 5$

$= 3x(2x - 5) + 1(2x - 5)$

$= (3x + 1)(2x - 5)$

$4x^2 - 20x + 25 = (2x)^2 + (5)^2 - 2(2x)(5)$

$= (2x - 5)^2$

$= (2x - 5)(2x - 5)$

L.C.M $= (2x - 5)^2 \times 3(2x + 5)(3x+1)$

$= 3(2x - 5)^2(2x+5)(3x+1)$

Q9. If H.C.F of $x^4+3x^3+5x^2+26x+56$ and $x^4 + 2x^3 - 4x^2 - x + 28$ is $x^2 + 5x + 7$, find the

Solution:

$$\textbf{L.C.M} = \frac{(x^4+3x^3+5x^2+26x+56)(x^4+2x^3-4x^2-x+28)}{x^2+5x+7}$$

$$x^2-2x+8$$

$$x^2+5x+7 \quad \cancel{x^4}+3x^3+5x^2+26x+56$$

$$-\cancel{x^4} \pm 5x^3 \pm 7x^2$$

$$-2\cancel{x^3}-2x^2+26x+56$$

$$\mp\cancel{2x^3} \mp 10x^2 \mp 14x$$

$$8x^2+40x+56$$

$$-8x^2 \pm 40x \pm 56$$

$$\times$$

L.C.M

$$= \left(x^2-2x+8\right)\left(x^4+2x^3-4x^2-x+28\right)$$

Q10. Simplify

(i) $$\frac{3}{x^3+x^2+x+1} - \frac{3}{x^3-x^2+x-1}$$

$$\frac{3}{(x^2+1)(x+1)} - \frac{3}{(x^2+1)(x-1)}$$

$$= \frac{3(x-1)-3(x+1)}{(x^2+1)(x+1)(x-1)}$$

$$= \frac{\cancel{3x}-3-\cancel{3x}-3}{(x^2+1)(x+1)(x-1)}$$

$$= \frac{-6}{(x^2+1)(x+1)(x-1)}$$

$$= \frac{-6}{(x^2+1)(x^2-1)}$$

$$= \frac{-6}{x^4-1} = \frac{6}{1-x^4} \text{ Ans.}$$

(ii) $\dfrac{a+b}{a^2-b^2} \div \dfrac{a^2-ab}{a^2-2ab+b^2}$

$$= \frac{a+b}{(a-b)(a+b)} \div \frac{a(a-b)}{(a-b)^2}$$

$$= \frac{1}{a-b} \div \frac{a}{a-b}$$

$$= \frac{1}{\cancel{a-b}} \times \frac{\cancel{a-b}}{a}$$

$$= \frac{1}{a}$$

Q11. Find square root by using factorization

$\left(x^2+\dfrac{1}{x^2}\right)+10\left(x+\dfrac{1}{x}\right)+27 \qquad (x \neq 0)$

Solution:

$$= \left(x^2+\frac{1}{x^2}\right)+10\left(x+\frac{1}{x}\right)+25+2$$

$$= x^2+\frac{1}{x^2}+2+10\left(x+\frac{1}{2}\right)+25$$

$$= \left(x+\frac{1}{x}\right)^2+10\left(x+\frac{1}{x}\right)+25$$

Let $x+\dfrac{1}{x}=a$

$$= a^2+10a+25$$

$$= (a+5)^2$$

Taking square root

$$= \sqrt{\left[\pm(a+5)\right]^2}$$

$$= \pm(a+5)$$

$$= \pm\left(x+\frac{1}{x}+5\right)$$

Q12. Find square root by using division method.

$\dfrac{4x^2}{y^2}+\dfrac{20x}{y}+13-\dfrac{30y}{x}+\dfrac{9y^2}{x^2} \quad (x, y \neq 0)$

Solution:

$$\begin{array}{r|l}
 & \frac{2x}{y}+5-\frac{3y}{x} \\
\hline
\frac{2x}{y} & \frac{4x^2}{y^2}+\frac{20x}{y}+13-\frac{30y}{x}+\frac{9y^2}{x^2} \\
 & -\frac{4x^2}{y^2} \\
\hline
\frac{4x}{y}+5 & \frac{20}{y}x+13 \\
 & -\frac{20}{y}x\pm 25 \\
\hline
\frac{4x}{y}+10-\frac{3y}{x} & -12-\frac{30y}{x}+\frac{9y^2}{x^2} \\
 & \mp 12 \mp \frac{30y}{x} \pm \frac{9y^2}{x^2} \\
\hline
 & \times
\end{array}$$

Required square root $= \pm\left(\dfrac{2x}{y}+5-\dfrac{3y}{x}\right)$

Objective

1. H.C.F of p^3q-pq^3 and $p^5q^2-p^2q^5$ is ___

(a) $pq(p^2-q^2)$ (b) $pq(p-q)$

(c) $p^2q^2(p-q)$ (d) $pq(p^3-q^3)$

2. H.C.F. of $5x^2y^2$ and $20x^3y^3$ is:___

(a) $5x^2y^2$ (b) $20x^3y^3$

(c) $100x^5y^5$ (d) $5xy$

3. H.C.F of $x-2$ and x^2+x-6 is ___

(a) x^2+x-6 (b) $x+2$

(c) $x-2$ (d) $x+2$

4. H.C.F of a^3+b^3 and a^2-ab+b^2 is ___

(a) $a+b$

(b) a^2-ab+b^2

(c) $(a-b)^2$ (d) a^2+b^2

5. H.C.F of x^2-5x+6 and x^2-x-6 is __:

(a) $x-3$ (b) $x+2$

(c) x^2-4 (d) $x-2$

6. H.C.F of a^2-b^2 and a^3-b^3 is___

(a) $a-b$ (b) $a+b$

(c) a^2+ab+b^2 (d) a^2-ab+b^2

7. H.C.F of x^2+3x+2, x^2+4x+3, x^2+5x+4 is:

(a) $x+1$ (b) $(x+1)(x+2)$

(c) $(x+3)$ (d) $(x+4)(x+1)$

8. L.C.M of $15x^2$, $45xy$ and $30xyz$ is__

(a) $90xyz$ (b) $90x^2yz$

(c) $15xyz$ (d) $15x^2yz$

9. L.C.M of a^2+b^2 and a^4-b^4 is:__

(a) a^2+b^2 (b) a^2-b^2

(c) a^4-b^4 (d) $a-b$

10. The product of two algebraic expression is equal to the ___ of their H.C.F and L.C.M.

(a) Sum

(b) Difference

(c) Product

(d) Quotient

11. Simplify $\frac{a}{9a^2-b^2}+\frac{1}{3a-b}=$ ___

(a) $\frac{4a}{9a^2-b^2}$

(b) $\frac{4a-b}{9a^2-b^2}$

(c) $\frac{4a+b}{9a^2-b^2}$

(d) $\frac{b}{9a^2-b^2}$

12. Simplify $\frac{a^2+5a-14}{a^2-3a-18}\times\frac{a+3}{a-2}=$ ___

(a) $\frac{a+7}{a-6}$ (b) $\frac{a+7}{a-2}$

(c) $\frac{a+3}{a-6}$ (d) $\frac{a-3}{a+2}$

13. Simplify

$$\frac{a^3-b^3}{a^4-b^4}\div\left(\frac{a^2+ab+b^2}{a^2+b^2}\right)=\_\_\_$$

(a) $\frac{1}{a+b}$ (b) $\frac{1}{a-b}$

(c) $\frac{a-b}{a^2+b^2}$ (d) $\frac{a+b}{a^2+b^2}$

14. Simplify :

$$\left(\frac{2x+y}{x+y}-1\right)\div\left(1-\frac{x}{x+y}\right)$$

=____

(a) $\frac{x}{x+y}$ (b) $\frac{x}{x-y}$

(c) $\frac{y}{x}$ (d) $\frac{x}{y}$

15. The square root of $a^2 - 2a + 1$ is __

(a) $\pm(a+1)$ (b) $\pm(a-1)$

(c) $a-1$ (d) $a+1$

16. What should be added to complete the square of $x^4 + 64$?

(a) $8x^2$ (b) $-8x^2$

(c) $16x^2$ (d) $4x^2$

17. The square root of $x^4 + \frac{1}{x^4} + 2$ is

__

(a) $\pm\left(x + \frac{1}{x}\right)$ (b) $\pm\left(x^2 + \frac{1}{x^2}\right)$

(c) $\pm\left(x - \frac{1}{x}\right)$ (d) $\pm\left(x^2 - \frac{1}{x^2}\right)$

18. The square root of $4x^2 - 12x + 9$ is:

(a) $\pm(2x - 3)$

(b) $\pm(2x + 3)$

(c) $(2x + 3)^2$

(d) $(2x - 3)^2$

19. L.C.M = ___

(a) $\frac{p(x) \times q(x)}{H.C.F}$ (b) $\frac{p(x).q(x)}{L.C.M}$

(c) $\frac{p(x)}{q(x) \times H.C.F}$ (d) $\frac{q(x)}{p(x) \times H.C.F}$

20. H.C.F. = ___

(a) $\frac{p(x) \times q(x)}{L.C.M}$ (b) $\frac{p(x) \times q(x)}{H.C.F}$

(c) $\frac{p(x)}{q(x) \times L.C.M}$ (d) $\frac{L.C.M}{p(x) \times q(x)}$

21. L.C.M x HCF=

(a) $p(x) \times q(x)$ (b) $p(x) \times H.C.F$

(c) $q(x) \times L.C.M$ (d) None

22. Any unknown expression may be found if ____ of them are known by using the relation L.C.M x H.C.F = p(x) x q(x)

(a) Two

(b) Three

(c) Four

(d) None

ANSWER KEY

1.	a	**2.**	a	**3.**	c	**4.**	b	**5.**	a
6.	a	**7.**	a	**8.**	b	**9.**	c	**10.**	c
11.	c	**12.**	a	**13.**	a	**14.**	d	**15.**	b
16.	c	**17.**	b	**18.**	a	**19.**	a	**20.**	a
21.	a	**22.**	b						